Superintegrability in the Manev Problem and its Real Form Dynamics

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Abstract

We report here the existence of Ermanno-Bernoulli type invariants for the Manev model dynamics which may be viewed upon as remnants of Laplace-Runge-Lenz vector whose conservation is characteristic of the Kepler model. If the orbits are bounded these invariants exist only when a certain rationality condition is met and thus we have superintegrability only on a subset of initial values. We analyze real form dynamics of the Manev model and derive that it is always superintegrable. We also discuss the symmetry algebras of the Manev model and its real Hamiltonian form.

1 Introduction

Since time immemorial the circular motion was the archetype motion of the heavenly bodies, and circle was assumed to be embodiment of perfection—in the East and the West alike. No wonder that when the observation data challenged this view a superposition of several circular motions or circular motion with an off-centre Sun were proposed in order to keep the circular paradigm intact.

Since Kepler and Newton elliptical trajectories replaced circular ones as an archetype of the (bounded) planetary motion and the circle is nowadays viewed upon rather as a degenerate ellipse than as an ideal incarnated. The archetype of elliptical motion is even exported to the atomic realm as we see depictions of 'the atom' with ellipses representing electrons' motion around the nuclei.

The advent of Einstein's theory did not produce a new archetype of heavenly motions, apart from the exceptional case of a collapse into the (still hypothetical) black holes. Nevertheless, among the variety of relativistic effects the perihelion advance of inner planets is definitely the best recognizable effect in the Solar system. Maybe it is time to accept a new archetype: *precessing ellipse* (or more generally, precessing conics). There are also classical arguments in its favour: Kepler-type motion is generally not preserved by small perturbations

and generally any sort of 'real world' interactions like Solar pressure, drag etc would destroy 'fixed ellipse' motion [1]. If precessing conics give us 'the typical' motion of planets it is tempting to ask which central force field produces them. Surprisingly or not, the answer is: *the Manev model* [2]. Here we already have persistent KAM tori and cylinders for a large class of even non Hamiltonian perturbations [1] and this is an additional argument in favour of the Manev model.

Kepler problem is famous as one of archetypes of superintegrable systems and probably, the first one where an unexpected non-Nöther symmetry has been uncovered. It is intriguing to ask whether Manev problem shares this property and here we report that this is indeed the case, but not for all initial data. Artificial models presenting such a behaviour are already known, e.g. sum of squares of Hamiltonians of independent oscillators, but Manev model is a 'real world' example having this property. Let's remark that for a generic central potential we could have disjoint set of initial data corresponding to closed orbits but in our case *all* points on certain level sets of the angular momentum lie on closed orbits which are intersections with the level sets of the additional invariant.

Also we will show that real form dynamics of the Manev problem—a closely connected dynamical model which we shall introduce below—is superintegrable for *all* initial data. Real form dynamics of the Manev problem is interesting enough to validate a separate analysis and we will describe it at the end of the article.

2 The Manev Problem Basics

By Manev model [3] we mean here the dynamics given by the Hamiltonian:

$$H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) - \frac{A}{r} - \frac{B}{r^2} \tag{1}$$

where $r = \sqrt{x^2 + y^2 + z^2}$; A and B are assumed to be arbitrary real constants whose positive values correspond to attractive forces. The genuine model proposed by George Manev was not invented as an approximation of relativity theory but as a consequence of Max Planck's (more general) action-reaction principle and it derived a specific value for the constant $B = \frac{3G}{2c^2}A$. Nevertheless, Manev model offers a surprisingly good practical approximation to Einstein's relativistic dynamics—at least at a solar system level—capable to describe both the perihelion advance of the inner planets and the Moon's perigee motion. In the last decade it had enjoyed an increased interest either as a very suitable approximation from astronomers' point of view or as a toy model for applying different techniques of the modern mechanics (see e.g. [4, 5, 6, 7, 8]).

Due to Hamiltonian's rotational invariance each component of the angular momentum

$$L_j = \varepsilon_{jkm}\, p_k x_m \qquad \{L_j\,, L_k\} = \varepsilon_{jkm}\, L_m \qquad \text{with} \;\; (x_1, x_2, x_3) = (x, y, z) \tag{2}$$

is an obvious first integral: $\{H, L_j\} = 0$ and so, like the Kepler problem, the Manev model is integrable. Components themselves are not in involution but span an $so(3)$ algebra with respect to the Poisson brackets.

The dynamics is confined on a plane which we assume to be Oxy and is separable in radial coordinates. On the reduced phase space (see e.g. [9] for the generalities of the reduction procedure) obtained by fixing the angular momentum $L_z \equiv L$ to a certain value ℓ the motion is governed by:

$$H_{\text{eff}} = \frac{1}{2}\left(p_r^2 + \frac{\ell^2 - 2B}{r^2}\right) - \frac{A}{r}. \tag{3}$$

The dynamics behave like radial motion of Kepler dynamics with angular momentum squared $\ell^2 - 2B$; while the case $2B > \ell^2$ corresponds to overall centripetal effect. On the other hand, the angular equation of motion $\dot{\theta} = L/r^2$ is still governed by the 'authentic' angular momentum ℓ (and r is as just described). Consequently, the remarkable properties of Kepler dynamics that all negative energy orbits are closed and the frequencies of radial and angular motions coincide (for any initial conditions) are no more true. Thus we may have not only purely classical perihelion shifts but also if $2B \geq \ell^2 \neq 0$ we may have collapsing trajectories which are spirals, even though in phase space they are symplectic transformations; while in the Kepler dynamics the only allowed fall down is along straight lines. For this reason the set of initial data leading to collision has a positive measure and this may offer an explanation why collisions in the solar system are estimated to happen more often than Newton theory predicts [10].

The dynamics of Manev model has already been thoroughly analyzed (see e.g. [4, 5, 6, 7, 8]) and we shall concentrate here on some of its invariants and resulting symmetry algebra.

3 The Kepler Problem Invariants

In the case of Kepler problem, corresponding to $B = 0$, we have more first integrals (for details and historical notes see e.g. [11, 12, 13, 14]):

$$J_x = p_y L + \frac{A}{r}x\,, \qquad J_y = -p_x L + \frac{A}{r}y\,, \qquad \{H_K, \vec{J}\} = 0. \tag{4}$$

where H_K is the Kepler Hamiltonian and J_x and J_y are the components of the Laplace-Runge-Lenz vector. They are not independent since

$$J^2 = 2H_K L^2 + A^2. \tag{5}$$

Together with the Hamiltonian and angular momentum they close on an algebra with respect to the Poisson brackets:

$$\{H_K, L\} = 0\,, \;\; \{L, J_x\} = J_y\,, \;\; \{L, J_y\} = -J_x\,, \;\; \{J_x, J_y\} = -2H_K L. \tag{6}$$

After redefining $\vec{E} = \vec{J}/\sqrt{|-2H_K|}$ we get:

$$\{L, E_x\} = E_y\,, \qquad \{L, E_y\} = -E_x\,, \qquad \{E_x, E_y\} = -\mathrm{sign}(H_K)\, L \tag{7}$$

which makes obvious the fact that we have an $so(3)$ algebra for negative energies and $so(2,1)$ for positive ones. In the case of the 3-dimensional Kepler problem the components of the angular momentum give us another copy of $so(3)$, see eq. (2), so the full symmetry algebra is $so(4)$ or $so(3,1)$ depending on the sign of H_K.

According to [12], the first use of these first integrals was made by J. Hermann (= J. Ermanno) in 1710 (in order to find *all* possible orbits under an inverse square law force) in the disguise of 'Ermanno-Bernoulli' constants:

$$J_\pm = J_x \mp \mathrm{i} J_y = \left(\frac{L^2}{r} - A \mp \mathrm{i}\, L p_r\right) \mathrm{e}^{\pm \mathrm{i}\theta} \tag{8}$$

which satisfy:

$$\{H_K, J_\pm\} = 0, \qquad \{L, J_\pm\} = \pm \mathrm{i} J_\pm, \qquad \{J_+, J_-\} = -4\mathrm{i} H_K L. \tag{9}$$

Curiously enough, the initial Kepler idea for a circular orbits with an off-centre Sun happens to be correct in a different context. As discovered by Hamilton [16], the *velocity* vector in the Kepler dynamics moves along a circle laying in a plane containing the origin but, in general, not centered, at the origin. If we choose the x-axis pointing to the point of closest approach, the centre of this circle is located at $(0, \epsilon A/\ell, 0)$ and its radius is A/ℓ where $\epsilon = \sqrt{1 + 2E\ell^2/A^2}$ is the eccentricity of the orbit. If the spatial orbit is hyperbola ($\epsilon > 1$) the velocity space orbit is only a section of the circle, otherwise the full circle is traversed.

4 The Manev Problem Invariants

Assuming $0 \neq \ell^2 > 2B$ and denoting

$$\nu^2 = \frac{\ell^2 - 2B}{\ell^2}\,, \qquad w = \frac{L^2}{r} - \frac{\ell^2}{\ell^2 - 2B} A \tag{10}$$

one easily verifies that

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left[\left(\nu w \pm \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\theta} w\right) \mathrm{e}^{\pm \mathrm{i}\nu\theta}\right] = 0 \tag{11}$$

and since $\frac{\mathrm{d}}{\mathrm{d}\theta} = \frac{r^2}{L}\frac{\mathrm{d}}{\mathrm{d}t}$ we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\nu \frac{L^2}{r} - \frac{A}{\nu} \mp \mathrm{i} L p_r\right) \mathrm{e}^{\pm \mathrm{i}\nu\theta}\right] = 0. \tag{12}$$

In the case when $\ell \neq 0,\ \ell^2 > 2B\,,\ H < 0$ and $A > 0$ the motion is on a 2-dimensional torus. In order to have globally defined constants of motion in this case we have to require that the ν's be rational i.e.

$$\nu = \sqrt{\ell^2 - 2B} : \ell = m : k \tag{13}$$

with m and k integers. Then due to eq. (11)

$$\mathcal{J}_\pm = \mathcal{J}^*_\mp = \left[\frac{m}{k}\frac{L^2}{r} - \frac{k}{m}A \mp \mathrm{i}\, Lp_r\right] \mathrm{e}^{\pm \mathrm{i}m\theta/\mathrm{k}} \tag{14}$$

are conserved by the flow of eq. (1) on a surface $L = \ell$ satisfying the rationality condition (13). Thus we have conditional constants of motion which exist only for disjoint but infinite set of values ℓ (c.f. invariant relations of [15]).

The trajectory in the configuration space is a 'rosette' with m petals and this is connected to the fact that $\mathcal{J}_\pm$ are invariant under the action of the cyclic group generated by rotations by angle $\frac{2\pi k}{m}$:

$$\theta \to \theta + 2\pi\frac{k}{m}n \qquad n = 0, 1, \ldots, m-1\,. \tag{15}$$

While in the Kepler case we could unambiguously attach the Laplace-Runge-Lenz vector to Ermanno-Bernoulli invariants this is not possible now due to this finite symmetry. (It is intuitively clear that if the Laplace-Runge-Lenz vector points to the perihelion of the Kepler ellipse, now we have m petals to choose between.) Anyway, up to this unambiguity, *or* restricting ourselves to one of the m sectors we may note that while the radial/angular components of the Laplace-Runge-Lenz vector take the form:

$$J_r = \frac{L^2}{r} - A \qquad J_\theta = -Lp_r \tag{16}$$

in our case $\mathcal{J}_r + \mathrm{i}\mathcal{J}_\theta = \left(\nu\frac{\ell^2}{r} - \frac{A}{\nu} - \mathrm{i}\ell p_r\right)\mathrm{e}^{\mathrm{i}(\nu-1)\theta}$ and hence:

$$\begin{aligned} \mathcal{J}_r &= \left(\nu\frac{\ell^2}{r} - \frac{A}{\nu}\right)\cos(\nu-1)\theta + \ell p_r \sin(\nu-1)\theta \\ \mathcal{J}_\theta &= -\ell p_r \cos(\nu-1)\theta + \left(\nu\frac{\ell^2}{r} - \frac{A}{\nu}\right)\sin(\nu-1)\theta\,. \end{aligned} \tag{17}$$

Turning to the algebraic properties of the new invariants one finds that the Poisson brackets between the real and imaginary parts of $\mathcal{J}_\pm = \mathcal{J}_0 \mp \mathrm{i}\mathcal{J}_1 = \mathcal{J}_x \mp \mathrm{i}\mathcal{J}_y$ are:

$$\begin{aligned} \{H, \mathcal{J}_{0,1}\} &= 0, \qquad \{L, \mathcal{J}_0\} = \frac{m}{k}\mathcal{J}_1, \qquad \{L, \mathcal{J}_1\} = -\frac{m}{k}\mathcal{J}_0 \\ \{\mathcal{J}_0, \mathcal{J}_1\} &= -\frac{m}{k}L\left[2H - \frac{2B}{r^2}\left(\frac{2L^2}{\ell^2} - 1\right)\right]. \end{aligned} \tag{18}$$

Here we have a $1/r^2$ term which seems to obstruct the Poisson brackets to form a closed algebra. Fortunately, redefining $\mathcal{E}_{0,1} = \mathcal{J}_{0,1}/L\sqrt{L^2 - \ell^2}$ we obtain the closed algebra $\mathfrak{g}_{H,L}$:

$$\begin{aligned} \{H, \mathcal{E}_{0,1}\} &= 0, \qquad \{L, \mathcal{E}_0\} = \frac{m}{k}\mathcal{E}_1, \qquad \{L, \mathcal{E}_1\} = -\frac{m}{k}\mathcal{E}_0 \\ \{\mathcal{E}_0, \mathcal{E}_1\} &= \frac{m}{k}\frac{1}{(L^2 - \ell^2)^2}\left[2HL + \frac{k^2}{m^2}\frac{A^2}{L^3}\frac{2L^2 - \ell^2}{L^2 - \ell^2}\right] \end{aligned} \tag{19}$$

in which H is a central element and L, $\mathcal{E}_0$ and $\mathcal{E}_1$ can be viewed as Cartan and root-vector generators. Due to (19) $\mathfrak{g}_{H,L}$ is a deformation of $gl(2)$. Of course, we have in addition the $so(3)$ algebra (2).

Similarly, in the case when $0 \neq \ell^2 < 2B$ we may denote $\frac{2B-\ell^2}{\ell^2} = \upsilon^2$ with υ real and

$$\mathcal{E}_\pm = \left[\upsilon\frac{L^2}{r} + \frac{A}{\upsilon} \mp Lp_r\right] \frac{\mathrm{e}^{\mp\upsilon\theta}}{L\sqrt{L^2-\ell^2}} \tag{20}$$

are first integrals for any ℓ and they satisfy:

$$\begin{aligned} \{H, \mathcal{E}_\pm\} &= 0\,, \qquad \{L, \mathcal{E}_\pm\} = \mp\upsilon\mathcal{E}_\pm \\ \{\mathcal{E}_+, \mathcal{E}_-\} &= \frac{2\upsilon}{(L^2-\ell^2)^2}\left[2HL - \frac{A^2}{\upsilon^2 L^3}\frac{2L^2-\ell^2}{L^2-\ell^2}\right]. \end{aligned} \tag{21}$$

The algebra $\mathfrak{g}'_{H,L}$ satisfied by H, L and $\mathcal{E}_\pm$ is quite analogous to $\mathfrak{g}_{H,L}$ but with a different function at the right hand side of the bracket $\{\mathcal{E}_+, \mathcal{E}_-\}$.

Finally, when $\ell^2 = 2B$ we have the first integral:

$$j = Lp_r + A\theta \tag{22}$$

satisfying $\{H, j\} = 0$, $\{L, j\} = A$.

5 Real Form Dynamics

Here we briefly recall the notion of real form (RF) dynamics referring the reader to [17] for more details and a list of examples.

We start from a standard (real) Hamiltonian system $\mathcal{H} \equiv \{\mathcal{M}, \omega, H\}$ with n degrees of freedom and at the present stage we assume that our phase space is just a vector space $\mathcal{M} = \mathbb{R}^{2n}$.

Let's consider its complexification: $\mathcal{H}^{\mathbb{C}} \equiv \left\{\mathcal{M}^{\mathbb{C}}, H^{\mathbb{C}}, \omega^{\mathbb{C}}\right\}$ where $\mathcal{M}^{\mathbb{C}}$ can be viewed as a linear space over the field of complex numbers:

$$\mathcal{M}^{\mathbb{C}} = \mathcal{M} \oplus \mathrm{i}\mathcal{M}.$$

In other words the dynamical variables in $\mathcal{M}^{\mathbb{C}}$ now take complex values. We assume that the Hamiltonian H (as well as all other possible first integrals in involution I_k) are *real analytic functions* on $\mathcal{M}$ which can naturally be extended to $\mathcal{M}^{\mathbb{C}}$. We introduce on the phase space $\mathcal{M}$ an involutive, symplectic automorphism $\mathcal{C} : \mathcal{M} \to \mathcal{M}$:

$$\mathcal{C}^2 = \mathbb{1}\,, \qquad \mathcal{C}(\{F, G\}) = \{\mathcal{C}(F), \mathcal{C}(G)\} \tag{23}$$

where with some abuse of terminology we use the same notation for the action of $\mathcal{C}$ on the dual of the phase space.

Since $\mathcal{C}$ has eigenvalues 1 and -1, it naturally splits $\mathcal{M}$ into two eigenspaces:

$$\mathcal{M} = \mathcal{M}_+ \oplus \mathcal{M}_- \tag{24}$$

whose dimensions need not be equal. Due to the fact that $\mathcal{C}$ is symplectic $\mathcal{M}_-$ and $\mathcal{M}_+$ are symplectic subspaces of $\mathcal{M}$ and we will write $\omega = \omega_+ \oplus \omega_-$.

Assuming a symplectic frame adapted to $\mathcal{C}$ we have:

$$\omega = \sum_{k=1}^{n_+} \mathrm{d}p_{k+} \wedge \mathrm{d}q_{k+} + \sum_{k=1}^{n_-} \mathrm{d}p_{k-} \wedge \mathrm{d}q_{k-}.$$

The automorphism $\mathcal{C}$ can naturally be extended to $\mathcal{M}^{\mathbb{C}}$ and it splits it again into a direct sum of two eigenspaces:

$$\mathcal{M}^{\mathbb{C}} = \mathcal{M}_-^{\mathbb{C}} \oplus \mathcal{M}_+^{\mathbb{C}}.$$

Similarly, the action of the complex conjugation * produces splitting into real and imaginary parts of the corresponding spaces. By construction $\mathcal{C}$ commutes with $*$ and their composition $\widetilde{\mathcal{C}} \equiv \mathcal{C} \circ {}^* = {}^* \circ \mathcal{C}$ is also an involutive symplectic automorphism on $\mathcal{M}^{\mathbb{C}}$; then we define $\mathcal{M}_{\mathbb{R}}$ to be the fixed point set of $\widetilde{\mathcal{C}}$ i.e.

$$\mathcal{M}_{\mathbb{R}} = \mathsf{Re}\mathcal{M}_+^{\mathbb{C}} \oplus \mathsf{i}\,\mathsf{Im}\mathcal{M}_-^{\mathbb{C}}$$

and it is again a symplectic subspace. From now on we will be interested in dynamics on $\mathcal{M}_{\mathbb{R}}$ and its connection to the initial real dynamical system.

In order to construct 'real form dynamics' we shall assume that the Hamiltonian is $\mathcal{C}$-invariant, i.e.:

$$\mathcal{C}(H) = H. \tag{25}$$

Then the Hamiltonian on the complexified phase space $H^{\mathbb{C}}$ (being the same analytical function of the complexified variables) will share this property.

The *'real form dynamics'* may be defined either as:

i) complexified Hamilton equations on $\mathcal{M}^{\mathbb{C}}$ being consistently restricted to $\mathcal{M}_{\mathbb{R}}$. This gives a real vector field tangent to $\mathcal{M}_{\mathbb{R}}$ and satisfying the equations of motion given by the real part of $H^{\mathbb{C}}$ *or*

ii) dynamics on $\mathcal{M}_{\mathbb{R}}$ defined by the restricted $H^{\mathbb{C}}$ and $\omega^{\mathbb{C}}$ (whose restrictions are real on $\mathcal{M}_{\mathbb{R}}$):

$$\begin{aligned} H\big|_{\mathcal{M}_{\mathbb{R}}} &= \frac{H + \widetilde{\mathcal{C}}(H)}{2} = \frac{H + \mathcal{C}(H)^*}{2} = \mathsf{Re}H^{\mathbb{C}} \\ \omega^{\mathbb{C}}\big|_{\mathcal{M}_{\mathbb{R}}} &= \mathrm{d}\,\mathsf{Re}p_+^{\mathbb{C}} \wedge \mathrm{d}\,\mathsf{Re}q_+^{\mathbb{C}} - \mathrm{d}\,\mathsf{Im}p_-^{\mathbb{C}} \wedge \mathrm{d}\,\mathsf{Im}q_-^{\mathbb{C}}. \end{aligned} \tag{26}$$

Now we have a well defined dynamical system $\mathcal{H}_{\mathbb{R}} = \{\mathcal{M}_{\mathbb{R}},\ \omega|_{\mathcal{M}_{\mathbb{R}}},\ H|_{\mathcal{M}_{\mathbb{R}}}\}$ with *real Hamiltonian* and *real symplectic form* on a subspace of the complexified phase space.

It is noteworthy that the 'real form dynamics' corresponding to a Liouville integrable Hamiltonian system is Liouville integrable again [17]. Similarly, the 'real form dynamics' corresponding to a superintegrable Hamiltonian system is superintegrable again. In such a case we have $\kappa \in [n+1, 2n-1]$ independent constants of motion which are no more in involution. It could easily be checked that they will again produce κ independent constants of motion of the RF dynamics.

6 Real Form Dynamics of the Manev Problem

The Manev Hamiltonian (and the canonical symplectic form as well) is invariant under the involution $\mathcal{C}$ reflecting the y-degree of freedom:

$$\begin{array}{lll} \mathcal{C}(x) = x, & \mathcal{C}(y) = -y, & \mathcal{C}(z) = z \\ \mathcal{C}(p_x) = p_x, & \mathcal{C}(p_y) = -p_y, & \mathcal{C}(p_z) = p_z. \end{array} \tag{27}$$

Consequently, the 'real form dynamics' of Manev model for this choice of involution will be given by:

$$\begin{aligned} H_{\mathbb{R}} &= \frac{1}{2}(p_x^2 - p_y^2 + p_z^2) - \frac{A}{\rho} - \frac{B}{\rho^2} \\ \omega_{\mathbb{R}} &= \mathrm{d}p_x \wedge \mathrm{d}x - \mathrm{d}p_y \wedge \mathrm{d}y + \mathrm{d}p_z \wedge \mathrm{d}z \end{aligned} \tag{28}$$

where $\rho = \sqrt{x^2 - y^2 + z^2}$ is the 'radius' of the pseudo-sphere . This is not an ordinary central field dynamics but rather an 'indefinite metric central field' as $H_{\mathbb{R}}$ depends on indefinite metric distance ρ. The real form Hamiltonian $H_{\mathbb{R}}$ and the appropriate 'angular momentum' $\tilde{L}_j$ are still commuting first integrals and the model is integrable. The involution acts on $\tilde{L}_j$ according to: $\mathcal{C}(\tilde{L}_j) = (-1)^j \tilde{L}_j$ and

$$\{\tilde{L}_j\, , \tilde{L}_k\} = \varepsilon_{jki}(-1)^{j+k+1}\tilde{L}_i \tag{29}$$

instead of eq. (2); the corresponding algebra is $so(2,1)$ which is the real form of $so(3)$ obtained with a $\mathcal{C}$-induced Cartan involution.

We shall assume again that the motion is on the Oxy-plane and in order to avoid the question of the behavior of trajectories on the singularities we restrict our attention on the $\mathcal{C}$-invariant configuration space:

$$\{(x, y, z) \in \mathbb{R}^2 \mid z = 0, x^2 > y^2, x > 0\}.$$

Then the dynamics is separable in pseudo-radial coordinates $\vartheta = \operatorname{artanh}(y/x) \in (-\infty, \infty)$ and $\rho \in (0, \infty)$:

$$\begin{aligned} H &= \frac{1}{2}\left(p_\rho^2 - \frac{\pi_\vartheta^2}{\rho^2}\right) - \frac{A}{\rho} - \frac{B}{\rho^2} \\ \omega &= \mathrm{d}p_\rho \wedge \mathrm{d}\rho + \mathrm{d}\pi_\vartheta \wedge \mathrm{d}\vartheta \end{aligned} \tag{30}$$

with $\tilde{L} \equiv \tilde{L}_z = \pi_\vartheta$, hence $\dot{\pi_\vartheta} = 0$ and $\dot{\vartheta} = -\tilde{L}/\rho^2$. Due to the different symplectic form $\tilde{L}$ generates now transformations which preserve ρ.

The type of the ρ-trajectories could be easily read off after the observation that the value of the real form Hamiltonian will be:

$$h = \frac{1}{2}\left(\dot{\rho}^2 - \frac{\ell^2}{\rho^2}\right) - \frac{A}{\rho} - \frac{B}{\rho^2}$$

due to $\dot{x}^2 - \dot{y}^2 = \dot{\rho}^2 - \ell^2/\rho^2$ and denoting the value of $\tilde{L}$ by ℓ. Introducing $v = \dot{\rho}$ and $u = 1/\rho$ we obtain an equation describing conics in the (u, v)-space:

$$u^2(\ell^2 + 2B) + 2Au + (2h - v^2) = 0.$$

Performing the same type of analysis as in [5] we may conclude that we may have three types of qualitatively different dynamical regimes:

for $\ell^2 + 2B > 0$ we have a family of hyperbolas.

for $\ell^2 + 2B = 0$ we have a family of parabolas for $A \neq 0$ which degenerate at $A = 0$ into pair of lines parallel to the $1/\rho$-axis.

for $\ell^2 + 2B < 0$ (only possible for repulsive Manev term) we have a family of ellipses.

Of course, in all these cases we have to exclude the region $u < 0$.

In order to obtain more specific information about the motion we will need an equation for the trajectories. Let's note again that in the case of non-vanishing angular momentum we have $dt = -\frac{\rho^2}{\tilde{L}} d\vartheta$. As a result the equation for the ρ-motion takes the form:

$$\frac{\mathrm{d}^2}{\mathrm{d}\vartheta^2} \frac{\tilde{L}^2}{\rho} - \frac{\ell^2 + 2B}{\ell^2} \frac{\tilde{L}^2}{\rho} - A = 0. \tag{31}$$

Assuming $\ell^2 + 2B \neq 0$ we introduce

$$\upsilon^2 = \frac{\ell^2 + 2B}{\ell^2}, \qquad w = \frac{\tilde{L}^2}{\rho} + \frac{\ell^2}{\ell^2 + 2B} A \tag{32}$$

and obtain an inverted oscillator–type equation:

$$\frac{\mathrm{d}^2}{\mathrm{d}\vartheta^2} w - \upsilon^2 w = 0. \tag{33}$$

Denoting by c_j the integration constants below we conclude that:

If $\ell^2 + 2B > 0$ the solution $\rho^{-1}(\vartheta)$ will be:

$$\frac{\tilde{L}^2}{\rho} = c_1 \cosh(\upsilon\vartheta) + c_2 \sinh(\upsilon\vartheta) - \frac{A}{\upsilon^2}. \tag{34}$$

Trajectories may collapse ($\rho \to 0$) for $\vartheta \to \pm\infty$ and $c_1 > c_2 > 0$, *or* ρ may tend to ∞ as ϑ tends to certain values ϑ_{min} and ϑ_{max}.

If $\ell^2 + 2B = 0$ the solution of eq. (31) will be:

$$\rho^{-1} = \frac{A}{2\tilde{L}^2} \vartheta^2 + c_3 \vartheta + c_4\,. \tag{35}$$

If $A > 0$ trajectories collapse for $\vartheta \to \pm\infty$ and if $A < 0$ then $\rho \to \infty$ as ϑ tends to some ϑ_{min} and ϑ_{max}. The case $A = 0$ leads to linear solution $\rho^{-1}(\vartheta)$ and corresponds either to the motion along fixed ρ or to trajectory starting at $\rho = \infty$ and some value of ϑ and collapsing for $\vartheta \to \infty$ (or its reverse).

If $\ell^2 + 2B < 0$ we will have a solution which oscillates harmonically between some values ρ_{min} and ρ_{max}:

$$\frac{\tilde{L}^2}{\rho} = c_5 \cos(\upsilon\vartheta) + c_6 \sin(\upsilon\vartheta) - \frac{A}{\upsilon^2}. \tag{36}$$

The case when $\rho_{min} < 0$ means that 'acceptable' motions will be trajectories coming from $\rho = \infty$ at $\vartheta = \vartheta_{min}$ and going to $\rho = \infty$ for some $\vartheta = \vartheta_{max}$.

In the special case of vanishing angular momentum we have 1-dimensional motion along the ray $\vartheta =$ const. It may be oscillating between some ρ_{min} and ρ_{max} or heading to collapse, or to infinity.

It is worth noting that the only trajectories which are compact in the (x, y)-space are those collapsing at both their ends in the origin tangentially to the boundaries $x = \pm y$ and those oscillating on a line interval of $\vartheta =$ const. This is to be contrasted to the standard Manev or Kepler problems.

Obviously, when $B = 0$ we will obtain a real form dynamics of the Kepler model. In this case we have even fewer possibilities for compact trajectories as we could not have oscillations along the line of $\vartheta =$ const.

Since the motion is never on a 2-torus the additional first integrals are always globally defined for all initial data. When $0 \neq \ell^2 > -2B$ they take the form:

$$\mathcal{J}_\pm = \left[\upsilon \frac{\tilde{L}^2}{\rho} + \frac{A}{\upsilon} \pm \tilde{L} p_\rho \right] \mathrm{e}^{\mp \upsilon\vartheta}. \tag{37}$$

As we did earlier we can introduce the renormalized $\mathcal{E}_\pm = \mathcal{J}_\pm / \tilde{L}\sqrt{\tilde{L}^2 - \ell^2}$ and derive for them the following symmetry algebra $\mathfrak{g}'_{H,\tilde{L}}$

$$\begin{aligned} \{H_{\mathbb{R}}, \mathcal{E}_\pm\} &= 0, \qquad \{\tilde{L}, \mathcal{E}_\pm\} = \mp\upsilon\mathcal{E}_\pm \\ \{\mathcal{E}_+, \mathcal{E}_-\} &= \frac{2\upsilon}{(\tilde{L}^2 - \ell^2)^2} \left[2H_{\mathbb{R}}\tilde{L} - \frac{A^2}{\upsilon^2 \tilde{L}^3} \frac{2\tilde{L}^2 - \ell^2}{\tilde{L}^2 - \ell^2} \right]. \end{aligned} \tag{38}$$

Like in (19) above $\mathfrak{g}'_{H,\tilde{L}}$ is a deformation of $gl(2)$ having the same H, $\tilde{L}$ dependence in the right hand side of (38), though L and $\tilde{L}$ have different properties.

Note that the algebras $\mathfrak{g}_{H,L}$ and $\mathfrak{g}'_{H,\tilde{L}}$ seem very close, i.e. they do not change effectively when passing from one real Hamiltonian form to the other. The reason for this is the fact, that all its generators are invariant with respect to the involution $\mathcal{C}$. The situation changes when we consider the algebra satisfied by $\tilde{L}_j$, see eq. (29).

In the case when $0 \neq \ell^2 < -2B$ let $\nu^2 = \frac{-(\ell^2+2B)}{\ell^2}$ and then invariants are:

$$\mathcal{J}_\pm = \mathcal{J}_\mp^* = \mathcal{J}_0 \mp \mathrm{i}\mathcal{J}_1 = \left[\nu \frac{\tilde{L}^2}{\rho} - \frac{A}{\nu} \mp \mathrm{i}\, \tilde{L} p_\rho \right] \mathrm{e}^{\mp \mathrm{i}\nu\vartheta} \tag{39}$$

Redefining again: $\mathcal{E}_{0,1} = \mathcal{J}_{0,1}/\tilde{L}\sqrt{\tilde{L}^2 - \ell^2}$ we obtain the brackets:

$$\begin{aligned} \{H_{\mathbb{R}}, \mathcal{E}_{0,1}\} &= 0, \qquad \{\tilde{L}, \mathcal{E}_0\} = -\nu\mathcal{E}_1, \qquad \{\tilde{L}, \mathcal{E}_1\} = \nu\mathcal{E}_0 \\ \{\mathcal{E}_0, \mathcal{E}_1\} &= \frac{-\nu}{(\tilde{L}^2 - \ell^2)^2}\left[2H_{\mathbb{R}}\tilde{L} + \frac{A^2}{\nu^2\tilde{L}^3}\frac{2\tilde{L}^2 - \ell^2}{\tilde{L}^2 - \ell^2}\right]. \end{aligned} \tag{40}$$

Finally, when $\ell^2 = 2B$ we have the first integral:

$$j = \tilde{L}p_\rho - A\vartheta \tag{41}$$

satisfying $\{H, j\} = 0$, $\{\tilde{L}, j\} = -A$.

7 Conclusions

The discovered existence of Ermanno-Bernoulli type invariants strengthens our belief that Manev model has an exceptional position among the central field theories. Not only it provides a better description of the real motion of the heavenly bodies than Kepler model but to a large extent it shares its superintegrability—probably its most celebrated mathematical feature. As a result it provides also a testbed for analysing the intricate interplay between integrability and superintegrability—a testbed having the advantage of being realistic and intuitive interaction.

Also, from the viewpoint of a RF dynamics enthusiasts we see here a curious (and encouraging) example when the RF dynamics—exotic as it may be—behaves 'better' than the original problem as it is always superintegrable.

As a final remark we note that the new results reported here do not contradict existing classifications of superintegrable models (see e.g. [18]) where only first integrals which are low order polynomials in momenta are described.

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